Morse homology for semi-flows

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Abstract

Assume M is a closed smooth manifold of finite dimension. The theory of hyperbolic dynamical systems provides all tools needed to construct the Morse complex associated to downward gradient flows on M; see [14, 15]. The corresponding homology groups, called Morse homology, represent singular homology of M. All this even extends to gradient flows on Banach manifolds; see [1].

By formal analogy one expects that Morse homology associated to the downward L^2 gradient equation of the classical action functional on the free loop space ΛM represents singular homology of the (infinite dimensional) Hilbert manifold ΛM. However, in this case the downward gradient only generates a *semi*-flow on ΛM, called the heat flow. As a consequence the isomorphism constructed for flows in [1] is not avavilable. New ideas are needed to show that Morse homology and singular homology of the free loop space are naturally isomorphic. Our attempts to solve this problem resulted in the discovery [18] of a *backward* λ-Lemma for the (forward) heat flow. We show how the backward λ-Lemma and elements of Conley theory can be used to construct a celular filtration of ΛM whose celular filtration complex is precisely the Morse complex.

Lectures 1 and 2: Finite dimensional case

Throughout fix a closed Riemannian manifold M of dimension n. Now pick a Morse function $f : M \to \mathbb{R}$. Its critical points serve as generators of the Morse chain groups which are graded by the Morse index. Counting flow lines of the downward gradient flow of f between critical points of index difference one defines the Morse boundary operator. The corresponding homology groups represent singular homology of M.

In the first two lectures we will give a detailed account of this construction, because major steps are the same in the infinite dimensional setting of Lectures 3 and 4. For this reason we include an extensive overview of the construction of Morse homology in the appendix below.

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Lecture 3: Heat flow Morse homology

Consider the Morse function on the free loop space of M, that is the Hilbert manifold $\Lambda M := W^{1,2}(S^1, M)$, given by the classical action functional

$$\mathscr{S} = \mathscr{S}_{V,g} : \Lambda M \to \mathbb{R}, \qquad \gamma \mapsto \int_0^1 \tfrac{1}{2}|\dot\gamma(t)|^2 - V_t(\gamma(t))\, dt,$$

where $V : S^1 \times M \to \mathbb{R}$ is a (generic) smooth function called potential or perturbation and $V_t := V(t,\cdot)$. The critical points of $\mathscr{S}$ are the (perturbed) closed geodesics in M, that is the solutions $x : S^1 \to M$ to the ODE

$$-\nabla_t \dot x - \nabla V_t(x) = 0.$$

These geodesics generate the chain groups $\mathrm{CM}_*(\Lambda M, \mathscr{S}_{V,g})$ and their Morse index provides the grading. The L^2 gradient equation of $\mathscr{S}$ is the heat equation

$$\tfrac{d}{ds} u_s - \mathrm{grad}_{L^2} \mathscr{S}(u_s) = \partial_s u - \nabla_t \partial_t u - \nabla V_t(u) = 0, \qquad u_s := u(s,\cdot), \tag{1}$$

for smooth cylinders $u : \mathbb{R} \times S^1 \to M$. If the Morse-Smale condition is satisfied the boundary operator is defined by counting isolated heat flow trajectories between closed geodesics of Morse index difference one. The corresponding Morse homology groups $\mathrm{HM}_*(\Lambda M, \mathrm{grad}_{L^2} \mathscr{S}_{V,g})$, called heat flow homology, have been constructed in [16].

To carry out this construction we interpret the heat equation in the spirit of Floer homology as a (parabolic) PDE in M with asympotic boundary condition at $s = \pm\infty$ provided by two prescribed critical points x and y of the action functional. Note that, despite the naming, not even a semi-flow appears here at all.

Lecture 4: Hyperbolic dynamics and Morse filtrations for the heat (semi-)flow

It is well known that the Cauchy problem associated to the heat equation (1) for maps $[0,\infty) \to \Lambda M$, $s \mapsto u_s$, with prescribed initial value $u_0 = \gamma \in \Lambda M$ admits a unique solution. The corresponding (forward) semi-flow $\{\varphi_s\}_{s\geq 0}$ on ΛM is called heat flow. It enjoys backward uniqueness and, in case $\mathscr{S}$ is Morse, each semi-flow line converges to a closed geodesic, as $s \to \infty$. But in general, there is no backward flow. Thus existence of a backward λ-Lemma [18] might be slightly surprising. We sketch its proof, because the subsequent construction of a Morse filtration of ΛM not only is based on the backward λ-Lemma, but in fact caused its discovery.

In order to construct a cellular filtration of the loop space which in a certain sense mirrors the heat flow Morse complex, we will define a Conley pair for each critical point and apply the backward λ-Lemma [18]. The whole construction relies on replacing the (non-existing) backward flow by taking pre-images under the time-s-map φ_s for $s > 0$. The outcome is a natural isomorphism between heat flow Morse homology and singular homology of the free loop space of M.

Appendix: Morse homology in finite dimensions

Fix a closed manifold M of finite dimension n. Pick a Morse function $f: M \to \mathbb{R}$ and consider the set Crit of critical points x of f. The Morse index of x is the number $\mathrm{ind}(x)$ of negative eigenvalues of the Hessian of f at x, equivalently, the dimension of the corresponding negative eigenspace E_x. Consider the free abelian group whose generators $\langle x \rangle := (x, o_x)$ are given by the critical points x of Morse index k together with an orientation o_x of E_x. Imposing the relation $(x, o_x) + (x, \bar{o}_x) = 0$ in case of opposite orientations defines the Morse chain group $\mathrm{CM}_k(M, f)$.

Fix one orientation for each critical point x and denote the set of chosen orientations by Or. Firstly, this amounts to fixing a basis of CM_* which provides the identification

$$\mathrm{CM}_k(M, f) \simeq \bigoplus_{x \in \mathrm{Crit}_k} \mathbb{Z}\langle x \rangle =: \widehat{\mathrm{CM}}_k(M, f, \mathrm{Or}; \mathbb{Z}).$$

Secondly, this gives rise to a natural boundary operator on $\widehat{\mathrm{CM}}_*$.

Pick, in addition, a Riemannian metric on M and consider the downward gradient vector field $-\nabla f$ on M whose flow we denote by $\{\varphi_s\}_{s \in \mathbb{R}}$. For each critical point x its unstable manifold $W^u(x) := \{p \in M \mid \lim_{s \to -\infty} \varphi_s p = x\}$ is a contractible submanifold of M of dimension $k = \mathrm{ind}(x)$; similarly for the stable manifold $W^s(x)$. As E_x coincides with the tangent space $T_x W^u(x)$, which itself is equal to the orthogonal complement of $T_x W^s(x)$, our choice of orientation of E_x orients $W^u(x)$ and co-orients $W^s(x)$.

For generic metrics g all unstable and stable manifolds intersect transversally, that is the Morse-Smale condition holds generically. Thus the sets $\mathscr{M}_{xy} := W^u(x) \cap W^s(y)$ of connecting trajectories are oriented manifolds (M being orientable or not). The dimension is the Morse index difference. If the index difference is one, then there are only finitely many (geometrically distinct) connecting trajectories. Comparing their orientations with the downward flow provides the characteristic sign $n_u \in \{-1, +1\}$ for each of these so-called isolated trajectories u between x and y. Denote by $n(x, y)$ the (finite) sum of all these characteristic signs. Counting isolated flow lines with characteristic signs defines the Morse boundary operator given by

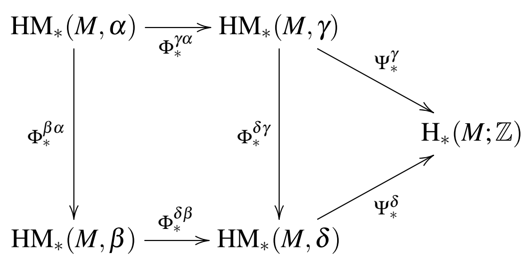
$$\partial_k = \partial_k(M, f, g, \mathrm{Or}): \widehat{\mathrm{CM}}_k \to \widehat{\mathrm{CM}}_{k-1}, \quad \langle x \rangle \mapsto \sum_{y \in \mathrm{Crit}_{k-1}} n(x, y) \langle y \rangle$$

for every x in the set Crit_k of critical points of index k. To prove that there are only finitely many isolated trajectories—thus $n(x, y)$ being well defined—requires to understand (non)compactness of the connecting manifolds $\mathscr{M}_{xy}$. To prove that $\partial^2 = 0$ amounts to understanding how to glue two connecting trajectories that meet (asymptotically) at a critical point. To solve these two problems we employ [14, 15] two fundamental tools of hyperbolic dynamics. Firstly, the Grobman-Hartman theorem [3, 4] takes care of the compactness issue. Secondly, the two λ-Lemmas of Palis [6] together take care of the gluing process; indeed the forward and the backward time λ-Lemma are used simultaneously.

Morse homology is the homology associated to the above chain complex, that is

$$\mathrm{HM}_*(M, f, g, \mathrm{Or}) := \mathrm{H}_*\left(\widehat{\mathrm{CM}}_*, \partial_*\right).$$

Abbreviating the auxiliary data (f,g,Or) by $\alpha=(f^\alpha,g^\alpha,\mathrm{Or}^\alpha)$ we will present Poźniak's construction [8] of isomorphisms among Morse homology associated to choices of auxiliary data α and β. These isomorphisms $\Phi_*^{\alpha\beta}$ are natural in the sense that the rectangular part of the diagram



commutes.

To prove that Morse homology actually represents singular homology there are various possibilities. For instance, Milnor [5] uses the freedom to change the Morse function. He picks a self-indexing one, that is $f(x)=k$ whenever x is a critical point of Morse index k, and constructs a Morse filtration of M (a cellular filtration which in a certain sense mirrors the Morse complex). In lecture four, in a more general infinite dimensional setting, we present a construction [17] which works for *any* Morse function. To achieve this we will use elements of Conley theory [2]. The resulting isomorphism renders the triangular part of the diagram above commutative.

For a construction of Morse homology in the spirit of Floer theory we recommend the book [9] by Schwarz; see [10] for an isomorphism to singular homology via pseudocycles.

Examples. Surfaces Σ in euclidean $\mathbb{R}^3$ equipped with (generic) height functions $f:\mathbb{R}^3\subset\Sigma\to\mathbb{R}$, $(x,y,z)\mapsto z$ and the induced metric are a rich source of examples where the Morse complex can be easily visualized.

a) Start with the unit sphere. Since there are no critical points of index difference one there are no isolated trajectories. Thus $\partial_k=0$, so homology is given by the chain groups (which are generated by the maximum and minium of f).

b) To obtain examples with nontrivial boundary operator deform the unit sphere such that a saddle point (Morse index one) appears. One could also replace spheres by tori.

Remark concerning history. The Morse complex goes back to the work of Thom [13], Smale [11, 12], and Milnor [5] in the 40's, 50's and 60's, respectively. The geometric formulation in terms of flow trajectories was re(dis)covered by Witten in his influential 1982 paper [19]. In fact, he studied a supersymmetric quantum mechanical system related to the Laplacian $\Delta_s=d_sd_s^*+d_s^*d_s$ which involves the deformed Hodge differential $d_s=e^{-sf}de^{sf}$ and acts on differential forms. Here $f:M\to\mathbb{R}$ denotes a Morse function and $s\geq 0$ is a real parameter. The geometric Morse complex arises as the adiabatic limit of the quantum mechanical system, as the parameter s tends to infinity. In the 90's the details of the construction of the geometric Morse complex have been worked out by Poźniak [8], Schwarz [9], and the author [14].

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